

Gravity theories, Transverse Doppler and Gravitational Redshifts in Galaxy Clusters

HongSheng Zhao,¹ John Peacock,² and Baojiu Li³

¹*Scottish University Physics Alliance, University of St Andrews, KY16 9SS, UK*

²*Scottish University Physics Alliance, University of Edinburgh, EH9 3JZ, UK*

³*Institute of Computational Cosmology, Department of Physics, Durham University, Durham DH1 3LE, UK*

(Dated: 1.1.2012; Email address: hz4@st-andrew.ac.uk)

There is growing interest in testing alternative gravity theories using the subtle Gravitational Redshifts in clusters of galaxies. However, current models all neglect a Transverse Doppler redshift of similar magnitude, and some models are not self-consistent. An equilibrium model would fix the Gravitational and Transverse Doppler velocity shifts to be about $6\sigma^2/c$ and $3\sigma^2/2c$ in order to fit the observed velocity dispersion σ self-consistently. This result is from the Virial Theorem for a spherical isotropic cluster, and is insensitive to the theory of gravity. In any case, a gravitational redshift signal cannot directly distinguish between the Einsteinian and $f(R)$ gravity theories, because the mass of the cluster dark halo must be treated as an unknown fitting parameter, whose value must vary according to the theory adopted, otherwise the system would be in equilibrium in one gravity theory and out of equilibrium in another.

PACS numbers: 98.10.+z, 95.35.+d, 98.62.Dm, 95.30.Sf

The theory of gravity has been subjected to various tests with the ever-improving quality of data from cosmology, galaxy clusters, galaxies and the solar system[1]. As shown by recent numerical N-body simulations on $f(R)$ type or scale-coupled gravities[2], dynamical data on non-linear cluster scales help to break theoretical degeneracies on linear cosmological scales, and overcome statistical uncertainties in observations. Past techniques often propose comparing lensing data and kinematic data with simulations [3], which often involve significant amount of modeling of the mass distribution before setting indirect constraints on the gravitational potential $\Phi(r)$ of the cluster. It would clearly be better to measure the gravitational potential in a galaxy cluster directly and compare this potential with the prediction from the Poisson equation for the mass distribution in a given gravity theory.

Indeed the gravitational potential is an observable. For example, spectral lines from the surface of the Sun are observed to be gravitationally redshifted by a small amount $GM_\odot/R_\odot c \sim 0.6 \text{ km s}^{-1}$, and the shift increases for more compact stars. On cosmic scales, the deepest potential well $\Phi(r)$ is in a cluster of galaxies with usually a bright central galaxy (BCG) near the bottom of the potential. A nearly spherical distribution of many hundreds of galaxies orbit around the BCG, with a Gaussian dispersion of random velocity of $\sigma(r) \sim 1000 \text{ km s}^{-1}$ in each direction. The observed line-of-sight Doppler shifts of galaxies relative to the BCG satisfy a Gaussian distribution with a small but non-zero mean velocity. This is partly due to the Gravitational Redshift (GR), a feature in any metric theory of gravity, caused by the spatial variation of the gravitational potential:

$$\Delta_{\text{GR}} = [\Phi_{\text{BCG}} - \Phi(r)]/c. \quad (1)$$

This signal of $\sim 10 \text{ km s}^{-1}$ becomes detectable above the σ/\sqrt{N} uncertainty of the mean velocity once the sample size $N \geq 10^4$. To obtain such a large sample for the first time, Wojtak et al. [4] use $N \sim 78000$ galaxies from about eight thousand clusters from the Sloan Digital Sky Survey (SDSS), divide the galaxies into four bins according to their projected distances R from their respective BCGs, "stack" their light-of-sight velocities relative to their BCGs, carefully remove interlopers, and finally compute the mean velocity in each bin. In this *Letter* we investigate the pros and cons of the gravitational redshift approach, and for the first time introduce a new effect in galaxy clusters.

In fact, the Gravitational Redshift is supplemented by an additional redshift of comparable amplitude. For any metric theory of gravity [1] the space time near a galaxy cluster is described by the metric $d\tau^2 = (1 + 2\Phi/c^2)dt^2 - (1 + z)^{-2}(1 + 2\Psi/c^2)dx^2$. Light emitted from a cluster at redshift z is time-dilated so that the ratio of the observed wavelength to the emitted wavelength changes as:

$$\frac{\lambda_{\text{obs}}}{\lambda_{\text{emit}}} = (1 + z) \left[1 + \frac{\Phi - \mathbf{v}^2/2}{c^2} \right], \quad (2)$$

so we have an additional shift

$$\Delta_{\text{TD}} = [\langle |\mathbf{v}|^2 \rangle - |\mathbf{v}_{\text{BCG}}|^2] / 2c, \quad (3)$$

owing to the Transverse Doppler (TD) effect from random motions of galaxies in special relativity (SR). Here we have neglected the much smaller shifts caused by the potential and motions within individual galaxies or stars. Wojtak et

al. [4] reported a blueshifting of the mean apparent line-of-sight velocity of the galaxies in the SDSS clusters, again relative to the BCG, which was then interpreted as GR. But this interpretation is incomplete. The TD effect always co-exists in proportion to GR because of the Virial Theorem:

$$\langle -\Phi/2 \rangle / 2c = \langle GM/r \rangle / 2c = \langle |\mathbf{v}|^2 \rangle / 2c \quad (4)$$

where $\langle \rangle$ denotes the averaging over all gravitational masses in the whole virialized volume of a cluster, and the factor of $1/2$ in front of Φ prevents double counting of the pairwise mutual potential. Thus the random kinetic energy per unit mass $\overline{\mathbf{v}^2}/2$ is globally 25% of the average potential $-\overline{\Phi}$. The ratio of $1/4$ holds even after averaging over a distribution of clusters of different mass and for clusters of any density profile and anisotropy parameter, so the Virial Theorem is a robust link between TD and GR effects, and their superposition is observed as the mean velocity shift.

Recent tests of general relativity are often posed in the context of the Λ Cold Dark Matter (CDM) model. It seems straightforward to test many gravity theories with their gravitational redshift prediction, but the invoking of dark haloes complicates the comparison of some gravity theories. E.g., it is well-known that in $f(R)$ gravity [7] the Newton constant G can be boosted by a factor which varies between 0 and $4/3$, depending on the environmental matter density. For a fixed cluster mass Wojtak et al. claimed that the $4/3 \simeq 1.33$ boost of the Gravitational Redshift signal in the Hu & Sawicki $f(R)$ model [5] with $|f_{R0}| = 10^{-4}$ robustly distinguishes Modified Gravity from Einsteinian Gravity. Such a claim, however, has a flaw: the total mass of the dark halo is an unknown free parameter, which must be determined by fitting the observed velocity dispersion as a function of distance from the cluster centre. Since G appears only in the combination GM , one can cancel essentially the enhancement of G in $f(R)$ gravity by reducing the halo mass parameter M , thus obtaining the *indistinguishable* fit to the velocity dispersion curve and to the mean velocity shift signal.

A more specific example is to use the hydrostatic equation

$$-\frac{d\Phi}{dr} = \frac{d(n\sigma^2)}{n dr} \quad (5)$$

For an isotropic number density of tracers (here galaxies) $n \propto r^{-\gamma}$ at large radius r for a Keplerian potential $\Phi = -GM/r$. This yields the solution

$$\frac{3}{2c}\sigma^2(r) = \frac{3}{2c}r^\gamma \int_r^\infty \frac{GM}{r^2} \frac{dr}{r^\gamma} = \frac{3}{2(\gamma+1)} \frac{GM}{rc}, \quad (6)$$

which implies that the random kinetic energy $\overline{\mathbf{v}^2}/2 = 3\sigma^2$ is locally $\sim 3/8 - 3/10$ of a Keplerian potential $|\Phi(r)|$ of a galaxy count profile with $\gamma \sim 3 - 4$ at large radii. The ratio $3/8$ or $3/10$ holds even after stacking of clusters of different masses and line-of-sight projection. This argument is true in standard gravity as well as in $f(R)$ gravity. Clearly gravity models with the same GM predict the same dispersion curve, and velocity shifts.

To compute the TD and GR effects generally at any projected radius, we start with the isotropic Jeans equation

$$-\frac{\partial(n\sigma^2)}{\partial Z} = n \frac{\partial\Phi}{\partial Z} \quad (7)$$

for the observable tracers (galaxies) with a number density $n(r)$ in equilibrium in the potential $\Phi(r)$. We integrate this equation over the line-of-sight depth Z through a cluster after multiplying by ZdZ on both side. Using the integration by parts, $Zd(n\sigma^2) = -dZ(n\sigma^2) + d(Zn\sigma^2)$, where the total derivative term vanishes at boundary, we find

$$\int_{-\infty}^{\infty} dZ (n\sigma^2) = \int_{-\infty}^{\infty} dZ \left(nZ \frac{\partial\Phi}{\partial Z} \right). \quad (8)$$

Multiplying by $2\pi R dR$ on both sides of the above equation predicts that the specific 3-dimensional kinetic energy averaged in an aperture of projected radius R to $R + dR$ is $\langle |\mathbf{v}|^2 \rangle / 2 = \langle QZ(\partial\Phi/\partial Z) \rangle = \langle GM(r)QZ^2r^{-3} \rangle$ inside the virial radius. Here $Q = 3/2$ from summing the three velocity components in quadrature. This equation allows us to predict the SR effect at all radii for any matter density in any metric-based gravity theory, since the Jeans equation applies to any force which is a gradient of a potential. E.g., if the density $\sim r^{-\gamma} \sim r^{-3}$ and gravity $\sim r^{-2}$ then

$$\frac{\langle v^2 \rangle}{2} = \frac{3}{8} \frac{\pi GM}{4R}. \quad (9)$$

Here the factor $\pi GM/4R$ comes from the density-weighted line-of-sight integration of $-\Phi(r)$.

To compute the GR and TD effects for the SDSS clusters, we follow the same methodology as Wojtak et al. We account for different cluster masses using a Salpeter-like mass function $dN/dM_{\text{vir}} \sim M_{\text{vir}}^{-2.33} \sim M_{\text{vir}}^{-7/3}$ taken between the mass range M_l and M_u .

$$\langle |\mathbf{v}|^2 \rangle / 2 = A(R)^{-1} \int_{M_l}^{M_u} dM_{\text{vir}} \int_{-\infty}^{\infty} dZ n(r) [Q Z^2 r^{-1} (d\Phi/dr)]|_{r=\sqrt{Z^2+R^2}}, \quad (10)$$

$$\langle \Phi \rangle = A(R)^{-1} \int_{M_l}^{M_u} dM_{\text{vir}} \int_{-\infty}^{\infty} dZ n(r) \Phi(r)|_{r=\sqrt{Z^2+R^2}}, \quad (11)$$

where $A(R) = \int_{M_l}^{M_u} dM_{\text{vir}} \int_{-\infty}^{\infty} dZ n(\sqrt{Z^2+R^2})$ is essentially a stacked line-of-sight integrated density at the projected radius R , and the spherical potential and tracer (galaxy) number count density are given by

$$\Phi(r) = -\frac{GM_{\text{vir}}}{rC_1} \ln(1 + rC/r_{\text{vir}}), \quad (12)$$

$$n(r) = \frac{BM_{\text{vir}}^{-7/3}}{4\pi C_1} \frac{N_{\text{vir}}}{r(r + r_{\text{vir}}/C)^2}, \quad (13)$$

where $C_1 \equiv [\ln(1+C) - \frac{C}{1+C}]$. We fix the halo concentration parameter $C = 5$ and the virial radius $r_{\text{vir}} = 1.2 (M_{\text{vir}}/10^{14} M_{\odot})^{1/3}$, as in Wojtak et al. Such a spherical NFW potential is an approximation to the true potential in Einsteinian gravity; in $f(R)$ gravity the potential from N-body simulations tends to be more concentrated [3] and in TeVeS gravity the potential tends to have a pure $\ln(r)$ profile at large radii. For the lack of a more precise analytical model of the tracer (galaxy) count $n(r)$, we have used the spherical NFW profile, and assumed the number of observable galaxies inside the virial radius to be $N_{\text{vir}} \sim M_{\text{vir}}/cst$. In reality, the number count profiles of observable bright galaxies tend to be more concentrated than the halo mass density. The selection criteria of galaxies of measurable redshift should be modeled as well, e.g., the preference for brighter and nearer galaxies and the velocity cut in the line of sight makes $n(R, Z)$ cylindrical instead of spherical, and $\int dM_{\text{vir}} \int_0^{60\text{Mpc}} 2dZ \int_0^{6\text{Mpc}} 2\pi R dR n(R, Z)$ is equal to the total size of the sample, about 78000 galaxies for Wojtak et al., which fixes the normalization constant B . Finally the BCGs are not always in the centre. We shall neglect these complexities, but empirically let $Q \simeq (3r_{\text{vir}} + r)/(2r_{\text{vir}} + r) \leq 3/2$ to mimic the effects of mild anisotropy and non-equilibrium at large radii (e.g. Fig. 1-3 of [8]).

The results for Einsteinian gravity is shown in Fig. 1, where we assume a halo mass range of $(M_l, M_u) = (0.11 \times 10^{15}, 2 \times 10^{15}) M_{\odot}$. Note that these are fitting parameters deduced from hydrostatic balancing of the pressure from the observable velocity dispersion σ and the halo gravity, as one cannot directly observe the halo and measure its mass. One can see our choice of parameters can fit the observed $(3\sigma_{\text{obs}}^2 - 3\sigma_{\text{BCG}}^2)/2c$ curve and, in doing so, we can predict the $-|\Phi(R) - \Phi_{\text{BCG}}|/c$ GR curve. Note the inevitable *reversal* from the observed average $\simeq 6.5 \pm 4 \text{ km s}^{-1}$ blueshifting to redshifting when within 0.2 Mpc of the BCG (cf. grey error bars and lines in Fig. 1) due to TD: the line-of-sight dispersion of non-BCGs $\sigma_{\text{obs}}(R) \simeq 600 \text{ km s}^{-1} \simeq 3\sigma_{\text{BCG}}$ converts directly to a $(3\sigma_{\text{obs}}^2 - 3\sigma_{\text{BCG}}^2)/2c \simeq 1.6 \text{ km s}^{-1}$ TD differential redshift near an isotropic cluster centre. The TD signal (red crosses) is clearly both non-negligible and model-insensitive, is thus a robust constraint applicable to any metric-based gravity theory.

As stated earlier, one should not compare an $f(R)$ gravity model with an Einsteinian gravity model for with the same halo mass distribution, namely $(M_l, M_u) = (0.11 \times 10^{15}, 2 \times 10^{15}) M_{\odot}$, since it would overpredict the velocity dispersion curve $\sigma^2(R)$ everywhere by the same factor of $4/3$, which can be ruled out even without measuring gravitational redshift. This is essentially because a unique relation between the SR and GR effect due to the isotropic Jeans equation. The more relevant mass distribution, where all (halo virial) mass is lowered by the same factor $4/3$, would predict a velocity dispersion curve identical as the Einsteinian curve. Instead, to show some difference, here we adopt $(M_l, M_u) = (0.09 \times 10^{15}, 1 \times 10^{15}) M_{\odot}$, and the result is shown as solid lines in Fig.1. This $f(R)$ model produces gravitational redshift and TD shifts by an amount essentially identical to Einsteinian gravity. So GR, TD and the velocity dispersion profile contain essentially three redundant copies of information about a metric theory, up to some uncertainty of anisotropy.

Likewise, the claimed $(0 - 10) \text{ km s}^{-1}$ extra shift in TeVeS reduces to only $(0 - 3) \text{ km s}^{-1}$ when [6] adopting mass models consistent with σ_{obs}^2 . Unfortunately the Transverse Doppler effect is left out explicitly in *all* these papers; e.g., TeVeS predicts a roughly radial-independent SR red-ward shift of $\langle \frac{3}{2c} Z \frac{\partial}{\partial Z} \Phi(r) \rangle = \frac{3\sigma_{\infty}^2}{\gamma c} \simeq 1 \text{ km s}^{-1}$ for $\gamma \sim 3$. Further

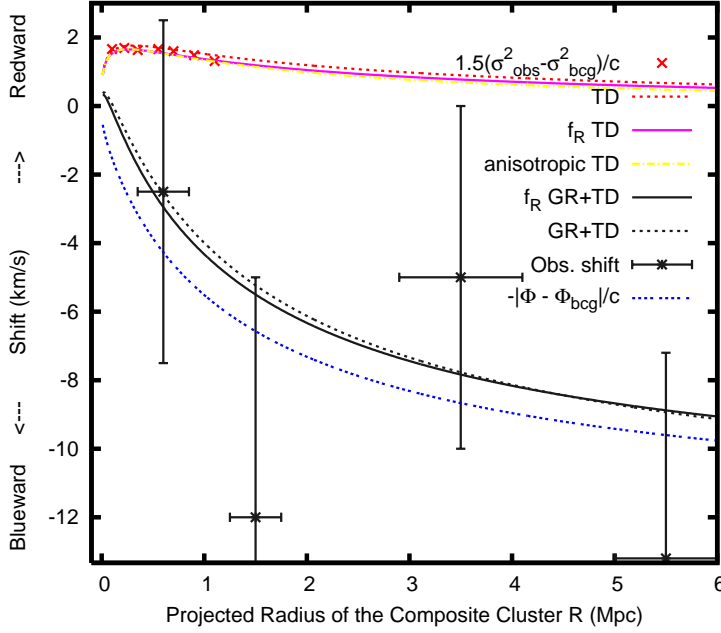


FIG. 1: Consistency of the data with the competing GR effect $-\overline{[\Phi(r) - \Phi_{\text{BCG}}]}/c$ plus TD effect $(\overline{\mathbf{v}^2} - \mathbf{v}_{\text{BCG}}^2)/2c$; the 3D kinetic energy $\overline{\mathbf{v}^2}/2$ at any R equals $\overline{QZ\partial\Phi/\partial Z}$ averaged over the line of sight depth z , where $Q = 3/2$ within the virial radius r_{vir} of an isotropic cluster; slight reduction of the TD signal (yellow line) if using a smaller $Q \simeq (3r_{\text{vir}} + r)/(2r_{\text{vir}} + r)$ to mimic any mild anisotropy and non-equilibrium at large radii ([8]). All models use the NFW halos weighted by $M^{-7/3}$ in the virial mass range $M = (0.09 - 1) \times 10^{15} M_{\odot}$ in $|f_{R0}| = 10^{-4}$ gravity (solid), where the effective gravitational constant (Fig. 3 of [7]) in these low mass halos is boosted by 33%, or $M = (0.11 - 2) \times 10^{15} M_{\odot}$ in Einstein gravity (dashed).

investigation including all relativistic effects and varying halo parameters perhaps within ranges from N-body $f(R)$ simulations [3, 7] is necessary before making a robust evaluation of gravity theories with this new probe.

Finally there is also the future possibility to use the spectra of cluster gas to measure the GR and TD effects for every X-ray cluster. The σ/\sqrt{N} uncertainty in the mean velocity will be negligible for the countless ionized gas particles, and there is less need for stacking different clusters. The signal will differ from the technique of Wojtak et al. because the X-ray gas particles have a profile different from that of galaxy number density, and less anisotropy than the velocity distribution of galaxies. By comparing the Transverse Doppler signals of different tracers one might even be able to infer the anisotropy of the motion of galaxies inside clusters.

-
- [1] Peacock J. A. *Cosmological Physics Cambridge University Press* (1999); Angus G. *et al.* On the proof of Dark Matter, the law of gravity and the mass of neutrinos *ApJ* **654** L13 (2007); Gentile G *et al.* Universality of galactic surface densities within one dark halo scale-length *Nature* **461**, 627 (2009); Gentile G. *et al.* Tidal dwarf galaxies as a test of fundamental physics *Astron. & Astrophys.* **472** L25 (2007); Galianni P., *et al.* Testing quasi-linear modified Newtonian dynamics in the solar system *arXiv 1111.6681* (2012);
 - [2] Li B. *et al.* Haloes and voids in $f(R)$ gravity *MNRAS* **421** 3481 (2012); Zhao H. *et al.* Structure Formation by Fifth Force: Power Spectrum from N-Body Simulations, *ApJ* **712** L179 (2010);
 - [3] Zhao G., Li B., Koyama K. Testing gravity using the environmental dependence of Dark Matter halos *Phys. Rev. Lett.* **107** 071303 (2011) Zhao G., Li B., Koyama K. N-body simulations for $f(R)$ gravity using Self-Adaptive Particle Mesh code *Phys. Rev. D* **83**, 044007 (2011)
 - [4] Wojtak R., *et al.* Gravitational redshift of galaxies in clusters as predicted by general relativity *Nature* **477**, 567 (2011)
 - [5] Hu, W., Sawicki I., Models of $f(R)$ Cosmic Acceleration that Evade Solar-System Tests *Phys. Rev. D* **76**, 064004 (2007)
 - [6] Bekenstein, J. D. , Sanders R.H., TeVeS/MOND is in harmony with gravitational redshifts in galaxy clusters, *arXiv:1110.5048* (2011)
 - [7] Schmidt, F. Dynamical masses in modified gravity. *Phys. Rev. D* **81**, 103002 (2010)
 - [8] Maccio A., *et al.* Mass of clusters in simulations *ApJ* **588**, 35-49 (2003)